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A THEOREM CONCERNING REAL FUNCTIONS.

By K. P. Williams.

The purpose of this note is to state a theorem that can be made to apply to any function not a constant defined on a linear interval, and which has a corollary one of the most important theorems concerning continuous functions. As the demonstration is very simple it is thought that the theorem can well be used in an introductory course on the real variable. While in such courses the Dirichlet definition of a function may be given, no general properties are usually obtained until some rather strong hypothesis, such as continuity, is imposed upon the function.

Let $f(x)$ be a function defined throughout the closed interval (a, b), and suppose it is not constantly of the same sign, and not everywhere zero. Then there exists at least one point such that in no vicinity of the point is the function constantly of the same sign.

Suppose the theorem not true. Then about each point of (a, b) we can construct an interval throughout which the function is of the same sign or zero. These intervals will overlap, so that by the Heine-Borel theorem a finite number of them covers (a, b). As these also overlap it follows that the function must be of the same sign throughout (a, b), or else everywhere zero, which is contrary to hypothesis. Thus the theorem must be true.

If $f(x)$ is continuous and both positive and negative, it follows at once that it is somewhere zero.

It is apparent that the theorem can be modified so as to generalize the theorem that a continuous function actually assumes all values between any two of its values. Let $f(x)$ be any function not constant, and M and m its upper and lower bounds, respectively. Then if l is any number between m and M there must be at least one point in no vicinity of which $l - f(x)$ is of the same sign.

The theorem can be generalized so as to apply to a function defined on a very general range. In the form that follows the terms used are those employed by Fréchet, "Sur Quelques Points du Calcul Fonctionel," Rendiconti del Circolo Matematico di Palermo, T. XXII, 1–42.

Let U be an operation defined for each element of an extremal class E, which is itself a sub-class of elements of a normal class. Then either there exists an element h of E such that on no sub-class of E to which h is properly interior is U constantly of the same sign, or else E can be broken up into a *finite number of closed sets, having no elements in common, and on each of which U is constantly of the same sign or constantly zero.*

Suppose there exists no point such as h. Then each element a of E gives rise to a sub-class E_a, to which a is properly interior and on which U is constantly of the same sign or constantly zero. From the generalized Heine-Borel theorem it follows that there is a finite number of these classes, call them E_{a_1}, E_{a_2}, $\cdots$, E_{a_m}, such that every element of E is interior to one of them. In case two of these classes have an element in common, they can be combined into a single class on which the function is constantly of the same sign or constantly zero. Suppose next that an element d of E_{a_i} is a limit element for elements of E_{a_j}. Since d is properly interior to some one of the classes we see that U is of the same sign or zero throughout the combined class $E_{a_i} + E_{a_j}$. It follows from this that the m sets $E_{a_1} E_{a_2} \cdots$, E_{a_m} can be replaced by n sets $(n < m)$, call them E_1, E_2, $\cdots$, E_n, which are all closed, have no elements in common, and on each of which U is constantly of the same sign or constantly zero. Thus the theorem is proved.

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